

# 2-MANIFOLD RECOGNITION IS IN LOGSPACE

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ABSTRACT. We prove that the homeomorphism problem for 2-manifolds can be decided in logspace. The proof relies on Reingold’s logspace solution to the undirected  $s, t$ -connectivity problem in graphs.

## 1. INTRODUCTION

Two compact, connected surfaces with (possibly empty) boundary are homeomorphic precisely when they have the same Euler characteristic, the same number of boundary components and they are either both orientable or both non-orientable. This triple of invariants leads to the classification theorem for compact surfaces, which has roots in work of Camille Jordan and August Möbius in the 1860s, and Max Dehn and Poul Heegaard in 1907, with the first rigorous proof due to Henry Brahana [5] in 1921 under the hypothesis that the compact surfaces are triangulated; a modern proof with this hypothesis is due to John Conway and presented by Francis and Weeks [8]. The fact that every compact surface has a triangulation, thus completing the classification theorem, was established by Tibor Radó [17] in 1925, and a modern proof using the “Kirby torus trick” was recently given by Allen Hatcher [10].

Once a solution to a decision problem has been found, it is natural to investigate its implementation and complexity. Lower dimensions are often used to get a foothold into higher dimensions—for instance, many algorithms for 3-dimensional manifolds follow Kneser’s blueprint and study them via surfaces embedded in them; the enumeration of triangulated 4-manifolds requires us to recognise the 3-dimensional sphere efficiently.

The main result of this paper asserts that given two finite 2-dimensional triangulations one can decide whether they represent homeomorphic surfaces using space logarithmic in the size of the triangulations. Letting  $L$  be the class of problems that can be decided in deterministic logspace, this is formally stated as:

**Theorem 1.1.** 2-MANIFOLD RECOGNITION *is in*  $L$ .

The definitions of “2-MANIFOLD RECOGNITION” and “*deterministic logspace*” are given in §2. We remark that while *a priori* a logspace algorithm has no time bound, it is easy to show that logspace algorithms run in polynomial time (see for example Lemma 4 in [7]).

There is a remarkable gap between surfaces and higher dimensional manifolds. Manifolds of dimension three were shown to be triangulated by Moise [15] in 1952, and an excellent discussion of the current state of the solution of the homeomorphism problem can be found in the recent survey by Matthias Aschenbrenner, Stefan Friedl and Henry Wilton [3]. In a nutshell, the homeomorphism problem for oriented 3-manifolds has been solved modulo the existence of an algorithm that determines whether an oriented 3-manifold has an orientation reversing involution. Many important algorithms for 3-manifolds have been implemented [6], and many important decision problems, such as unknot recognition [9] and 3-sphere recognition [19], have been shown to be in the complexity class NP.

The next qualitative gap arises between dimensions three and higher. There are compact 4-dimensional manifolds, such as the  $E_8$  manifold discovered by Mike Freedman in 1982, that are not homeomorphic to any simplicial complex. Ciprian Manolescu [13] has recently announced that there are also such examples in dimensions  $n \geq 5$ . Even if one restricts to compact, simplicial manifolds of dimension  $n \geq 4$ , the homeomorphism problem was shown to be undecidable by Markov [14] in 1958 as a consequence of the unsolvability of the isomorphism problem for finitely

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presented groups, which is due to Adyan [1, 2] and Rabin[16]. In particular, for the development of algorithms in higher dimensions one needs to restrict to special classes of manifolds, or else be content with heuristic methods.

We now give an informal description of our logspace algorithms. The starting point is a logspace algorithm which, given a single triangulation as input:

- (1) Checks that the triangulation is a 2-manifold.
- (2) Counts the number of connected components,  $c$ .
- (3) If the input is a connected 2-manifold:
  - decides if it is orientable or non-orientable;
  - computes the Euler characteristic,  $\chi$ ;
  - counts the number of boundary components,  $b$ .
- (4) If there is more than one connected component, the algorithm outputs the following data:  $(o_1, \chi_1, b_1), \dots, (o_c, \chi_c, b_c)$  where  $o_i = 0$  if the  $i$ -th connected component is orientable and 1 otherwise,  $\chi_i$  is its Euler characteristic, and  $b_i$  is its number of boundary components. Moreover, the algorithm outputs this data in the following order:
  - $o_i < o_{i+1}$ , or
  - $o_i = o_{i+1}$  and  $\chi_i < \chi_{i+1}$ , or
  - $o_i = o_{i+1}, \chi_i = \chi_{i+1}$ , and  $b_i \leq b_{i+1}$ .

This output is a complete invariant of the homeomorphism type of the 2-manifold, and so the solution to the homeomorphism problem then follows by running this algorithm simultaneously on two triangulations.

This paper is organised as follows. We give precise definitions of the complexity class and data structures we use in §2. The algorithms to verify that an input triangulation represents a surface and count the number of components are described in §3. The algorithm to compute the complete invariants of a connected, triangulated surface is given in §4, and this algorithm is then applied in §5 to compute the invariants of each connected component of a disconnected surface.

## 2. PRELIMINARIES

A *deterministic logspace transducer* consists of a finite state control, a read-head, and three tapes:

- (1) the *input tape* is read-only, and stores the input string;
- (2) the *work tape* is read-write, but is restricted to using at most  $c \log n$  squares, where  $n$  is the length of the word on the input tape and  $c$  is a fixed constant; and
- (3) the *output tape* is write-only, and is restricted to writing left to right only. The space used on the output tape is not added to the space bounds.

A transition of the transducer takes as input a letter of the input tape at the position of the read-head, a state of the finite state control, and a letter on the work-tape. On each transition the transducer can modify the work tape, change states, and write at most a fixed constant number of letters to the output tape, moving to the right along the output tape for each letter printed.

Since the position of the read-head of the input tape is an integer between 1 and  $n$  (the length of the input), we can store it in binary on the work tape. In addition we can store a finite number of *pointers* to positions on the input tape.

A problem is in deterministic logspace if it can be decided using a deterministic logspace transducer. Since all transducers in this article will be deterministic, we will say logspace for deterministic logspace throughout.

A key property of logspace transducers is that they can be composed together to give new logspace transducers. Formally, let  $X, Y$  be finite alphabets, and let  $X^*$  denote the set of all finite length strings in the letters of  $X$ . We call  $f : X^* \rightarrow Y^*$  a *logspace computable function* if there is a logspace transducer that on input  $w \in X^*$  computes  $f(w)$ .

**Lemma 2.1** (Lemma 2 in [7]). *If  $f, g : X^* \rightarrow X^*$  can both be computed in logspace, then their composition  $f \circ g : X^* \rightarrow X^*$  can also be computed in logspace.*

*Proof.* Let  $M_f, M_g$  be logspace transducers that compute  $f$  and  $g$  respectively. On input  $w \in X^*$ , run  $M_f$ . Each time  $M_f$  calls for the  $j$ th input letter, run  $M_g$  on  $w$ ; however, instead of writing the output of  $M_g$  to a tape, we add 1 to a counter (in binary) each time  $M_g$  would normally write a

letter. Continue running  $M_g$  until the counter has value  $j - 1$ , at which point we return the next letter that  $M_g$  would output back to  $M_f$ .  $\square$

Finally, a *logspace algorithm* is an algorithm that runs on a logspace transducer. Lemma 2.1 implies that a logspace algorithm may assume that its input is the output of some other logspace algorithm.

In this article we show that the following decision problem is in logspace:

*Problem:* 2-MANIFOLD RECOGNITION

*Instance:* Two 2-dimensional triangulations  $\mathcal{T}_1, \mathcal{T}_2$

*Question:* Do  $\mathcal{T}_1$  and  $\mathcal{T}_2$  represent homeomorphic 2-manifolds?

For a positive integer  $n$  let  $[n]$  denote the set  $\{1, \dots, n\}$ . A triangulation  $\mathcal{T}$  is specified by a list of  $n$  triangles, where each triangle  $t \in [n]$  has vertices labeled 1, 2, 3 (which induces an orientation on the triangle), and edges glued according to a table as follows:

	(12)	(23)	(31)
1	$a_1$	$b_1$	$c_1$
2	$a_2$	$b_2$	$c_2$
$\vdots$			
$n$	$a_n$	$b_n$	$c_n$

with

$$a_t, b_t, c_t \in \{\emptyset\} \cup \{(s, e) \mid s \in [n], e \in \{(12), (21), (23), (32), (31), (13)\}\}.$$

The entry  $a_t = (s, e)$  in row  $t$  column (12) means that the edge (12) in  $t$  is glued to the edge  $e$  in triangle  $s$ , whereas  $a_t = \emptyset$  means that the edge (12) in  $t$  is not glued to anything (i.e., it is a boundary edge); likewise for columns (23) and (31).

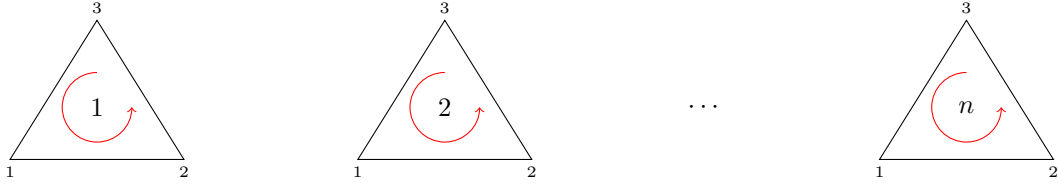


FIGURE 1. Input triangles

A triangulation is given to a logspace transducer by writing the string

$$\# a_1 b_1 c_1 \# a_2 b_2 c_2 \# \dots \# a_n b_n c_n$$

on the input tape using the alphabet  $\{\#, \emptyset, 0, 1, (12), (23), (31), (21), (32), (13)\}$  where  $a_i, b_i, c_i$  are written as either  $\emptyset$  or a binary number followed by (12), (23) or (31).

**Example 2.2.** Figure 2 illustrates a triangulation of a punctured Klein bottle, and Table 1 shows the corresponding table of edge gluings. For this triangulation, the input tape of our logspace transducer would read as follows:

$$\# 10 (13) 11 (12) 11 (32) \# 11 (13) \emptyset 1 (21) \# 1 (23) 1 (13) 10 (21)$$

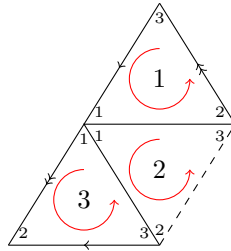


FIGURE 2. Triangles for Example 2.2 with edges identified. Dashed line is the boundary.

TABLE 1. Input table for Example 2.2

	(12)	(23)	(31)
1	2, (13)	3, (12)	3, (32)
2	3, (13)	$\emptyset$	1, (21)
3	1, (23)	1, (13)	2, (21)

A triangulation of  $n$  triangles has input size  $N \in O(n \log n)$ . We will prove that the data required to identify the homeomorphism type of the input can be output by a transducer using  $O(\log N)$  squares of the work tape. It follows that on input a triangulation with  $n$  triangles, the homeomorphism type can be computed using  $O(\log N) = O(\log n + \log \log n) = O(\log n)$  space.

It can easily be checked in logspace that the input is written in the required form – the number  $n$  of  $\#$  symbols can be computed and written in binary on the work tape by scanning  $\#$  symbols, then one can check that each binary number on the tape has value between 1 and  $n$ . So we may assume the input is correctly specified. However, we do not assume that the gluing instructions are consistent or give a manifold. For example, we may have  $\emptyset$  in row  $t$  column (12) but  $(t, (12))$  may appear as a different entry in the table, which would be inconsistent. The algorithm we describe will check this.

Note that the set  $\epsilon = \{(12), (21), (23), (32), (31), (13)\}$  comes naturally equipped with an involution  $\bar{\cdot} : \epsilon \rightarrow \epsilon$  given by  $\overline{(ij)} = (ji)$ .

When describing our algorithms we will refer to *row  $t$  column  $e$*  of the input tape, which means the entry in row  $t$  column  $e$  of the gluing table. This can be located in logspace by scanning the input tape from left to right counting the number of  $\#$  symbols.

Throughout this paper we make use of a deterministic logspace algorithm due to Reingold [18] which takes input  $(V, E, s, t)$ , where  $(V, E)$  is an undirected graph,  $s, t \in V$ , and returns *Yes* if there is an edge path from  $s$  to  $t$ , and *No* otherwise. We call this algorithm **REIN**.

We present the algorithms in this paper using pseudocode. Note that for-loops in the pseudocode are straightforward to implement a logspace transducer, using a binary number on the work tape for each loop. All of our algorithms make implicit use of the fact that logspace functions are closed under composition (Lemma 2.1).

### 3. INITIAL TESTS

**3.1. Counting components of a graph.** We begin with a simple tool that we call upon repeatedly in this paper: a logspace algorithm to compute the number of connected components of an undirected graph. This algorithm follows immediately from **REIN**. It operates as follows; see Algorithm 1 for the pseudocode.

Assume the graph is written on the input tape with vertices  $[n]$  and edges given as a list  $E \subseteq [n] \times [n]$ . Initialise a counter  $c = 1$  for the component containing vertex 1. The key idea is to iterate through the remaining vertices, and to increment  $c$  each time we encounter the lowest-numbered vertex of some connected component.

More precisely: Algorithm 1 runs through each vertex  $t > 1$ , calling **REIN** to test whether  $t$  is connected to any vertex  $s < t$ . If it is, we leave  $c$  unchanged and move to the next vertex. If it is not, we increment  $c$  and move to the next vertex.

#### Algorithm 1: Count connected components.

**Input:** Undirected graph  $([n], E)$ .  
**Output:** Number of connected components,  $c$ .  
Write a counter  $c = 1$  (in binary) to the work tape;  
**for**  $t = 2$  **to**  $n$  **do**  
    Set  $b = \text{false}$ ;  
    **for**  $s = 1$  **to**  $t - 1$  **do**  
        Run **REIN** on  $([n], E, s, t)$ . If **REIN** returns **true**, set  $b = \text{true}$ ;  
    **end**  
    If  $b = \text{false}$ , increment  $c$  by 1;  
**end**  
**return**  $c$ ;

**3.2. Checking the input is a surface.** Our first task is to test the validity of the input. Algorithm 2 decides whether the input represents a surface by enumerating through  $(t, e)$  for each  $t \in [n]$ ,  $e \in \{(12), (23), (31)\}$  and checking:

- (1) that the entry  $(t, e)$  or  $(t, \bar{e})$  appears at most once in the table;
- (2) if row  $t$  column  $e$  of the table is  $\emptyset$ , that neither  $(t, e)$  nor  $(t, \bar{e})$  appear in the table;
- (3) if row  $t$  column  $e$  of the table is  $(s, f)$  for  $f \in \{(12), (23), (31)\}$ , that row  $s$  column  $f$  is  $(t, e)$ ;
- (4) if row  $t$  column  $e$  of the table is  $(s, f)$  for  $f \in \{(21), (32), (13)\}$ , that row  $s$  column  $\bar{f}$  is  $(t, \bar{e})$ ;
- (5) that row  $t$  column  $e$  of the table is not  $(t, e)$  or  $(t, \bar{e})$ .

**Algorithm 2:** Check surface.

**Input:** Triangulation data on input tape.  
**Output:** *Yes* if input is a surface, *No* otherwise.  
**for**  $t \in [n]$  **do**  
  **for**  $e \in \{(12), (23), (31)\}$  **do**  
    Write a counter  $c = 0$  to the work tape;  
    Scan the tape from left to right reading each entry  $(s, f)$ ;  
    If  $(s, f) \in \{(t, e), (t, \bar{e})\}$ , increment  $c$  by 1;  
    If  $c > 1$ , output *No* and stop;  
    Read the entry  $y = (s, f)$  in row  $t$  column  $e$ ;  
    If  $y = \emptyset$  and  $c \neq 0$ , output *No* and stop;  
    If  $f \in \{(12), (23), (31)\}$ , read the entry  $z$  in row  $s$  column  $f$ . If  $z \neq (t, e)$ , output *No* and stop;  
    If  $f \in \{(21), (32), (13)\}$ , read the entry  $z$  in row  $s$  column  $\bar{f}$ . If  $z \neq (t, \bar{e})$ , output *No* and stop;  
    If  $y \in \{(t, e), (t, \bar{e})\}$ , output *No* and stop;  
  **end**  
**end**  
If *No* not printed, return *Yes*;

**3.3. Counting the number of connected components.** Next, we count the number of connected components of the input surface. To do this we construct the *face-dual graph* of the surface, which is an undirected graph whose vertices correspond to the triangles of the surface, and whose edges correspond to triangle gluings. More precisely, the vertices of the face-dual graph are  $[n]$ , and the edges of the face-dual graph are pairs  $(s, t)$  for which  $(t, e)$  is in row  $s$  of the table for some  $e \in \{(12), (23), (31), (21), (32), (13)\}$  (and therefore  $(s, e)$  appears in row  $t$  for some  $e$  also).

Note that the face-dual graph as defined here is a simple graph: it does not include loops or parallel edges (which do not affect connectivity). As an example, the face-dual graph for the punctured Klein bottle of Example 2.2 has vertices  $\{1, 2, 3\}$ , and edges  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{1, 3\}$ .

Algorithm 3 takes triangulation data as input and outputs the face-dual graph as an undirected graph  $([n], E')$ , using a simple scan through the table.

**Algorithm 3:** Construct the face-dual graph.

**Input:** Triangulation data on input tape.  
**Output:** Face-dual graph  $([n], E')$ .  
Scan the tape counting # symbols in binary on the work tape, then store this number  $n$  and write  $[n]$  to the output tape;  
**for**  $t \in [n]$  **do**  
  **for**  $s = t + 1, \dots, n$  **do**  
    Check whether  $(t, e)$  is in row  $s$  of the table for some  
     $e \in \{(12), (21), (23), (32), (31), (13)\}$ . If true, write  $(s, t)$  to the output tape;  
  **end**  
**end**

To count components of the input triangulation in logspace, we use Algorithm 3 to construct the face-dual graph, and we count components using Algorithm 1 with this face-dual graph as input. Call the composition of these algorithms *Algorithm A*.

#### 4. ALGORITHM FOR ONE CONNECTED COMPONENT

In this section we assume the input surface is connected and compute its homeomorphism type. In the next section we extend this to surfaces with more than one connected component.

**4.1. Orientability.** We can determine whether or not a manifold is orientable by taking its double cover, which is connected if the manifold is non-orientable, and which has two components if the manifold is orientable.

Recall that each triangle has a fixed orientation determined by the corner labels 1, 2, 3. The double cover is given by a set of  $2n$  triangles  $\{t, t' \mid t \in [n]\}$  with a gluing table constructed from the original table as follows:

- (1) the rows of the table are  $\{t, t' \mid t \in [n]\}$  and the columns are  $\{(12), (23), (31)\}$ ;
- (2) if row  $t$  column  $e$  of the original table contains  $\emptyset$ , then write  $\emptyset$  in rows  $t, t'$  column  $e$  of the new table;
- (3) if row  $t$  column  $e$  of the original table contains  $(s, f)$  with  $f \in \{(12), (23), (31)\}$ , then write  $(s', f)$  in row  $t$  column  $e$  and  $(s, f)$  in row  $t'$  column  $e$  of the new table;
- (4) if row  $t$  column  $e$  of the original table contains  $(s, f)$  with  $f \in \{(21), (32), (13)\}$ , then write  $(s, f)$  in row  $t$  column  $e$  and  $(s', f)$  in row  $t'$  column  $e$  of the new table.

Table 2 illustrates the double cover of the punctured Klein bottle from Example 2.2.

TABLE 2. Double cover data for Example 2.2

	(12)	(23)	(31)
1	2, (13)	3', (12)	3, (32)
2	3, (13)	$\emptyset$	1, (21)
3	1', (23)	1, (13)	2, (21)
1'	2', (13)	3, (12)	3', (32)
2'	3', (13)	$\emptyset$	1', (21)
3'	1, (23)	1', (13)	2', (21)

We can easily describe a logspace algorithm to produce this double cover gluing table from the original input; see Algorithm 4 for the details. To test the orientability of the original input triangulation, we now compose this with Algorithm A from Section 3.3: Algorithm 4 constructs the double cover, and Algorithm A tests whether the double cover has one or two components.

**4.2. Euler characteristic.** For a triangulation of a surface  $S$  we have  $\chi(S) = |V| - |E| + n$ , where  $V$  and  $E$  are the vertex set and edge set of  $S$  respectively, and where  $n$  is the number of triangles.

Let  $x$  be the number of edges of triangles that are not glued to any other edge, i.e., the number of  $\emptyset$  symbols on the input tape. Then the number of edges is  $|E| = (3n - x)/2 + x = (3n + x)/2$ , since the remaining  $3n - x$  triangle edges are identified in pairs. We can compute  $n$  and  $x$ , and hence  $|E|$ , in logspace by counting the number of  $\#$  and  $\emptyset$  symbols on the input tape.<sup>1</sup>

It remains to compute  $|V|$ . We do this by tracking the identifications of individual vertices of triangles. For this we construct an undirected graph  $K$ , which we call the *vertex identification graph*, as follows. The graph  $K$  has vertex set

$$W = \{w_{t,1}, w_{t,2}, w_{t,3} \mid t \in [n]\},$$

where  $w_{t,i}$  represents vertex  $i$  of triangle  $t$ . Note that the graph  $K$  has  $|W| = 3n$  vertices overall. In the punctured Klein bottle from Example 2.2, these vertices are

$$W = \{w_{1,1}, w_{1,2}, w_{1,3}, w_{2,1}, w_{2,2}, w_{2,3}, w_{3,1}, w_{3,2}, w_{3,3}\}.$$

The edge set of the graph  $K$  is  $F = \{(w_{t,i}, w_{s,j}) \mid w_{t,i}, w_{s,j} \text{ are identified directly}\}$ , where by “identified directly” we mean that some edge triangle  $t$  is glued to some edge of triangle  $s$  in a way

<sup>1</sup>Note that addition and division by two are both logspace computable.

**Algorithm 4:** Construct the double cover of a triangulation.**Input:** Triangulation data on input tape.**Output:** Triangulation data for the double cover, with triangles ordered as  $1, \dots, n, 1', \dots, n'$ .

```

for  $t \in [n]$  do
  Write # to the output tape;
  for  $e \in \{(12), (23), (31)\}$  do
    If row  $t$  column  $e$  of the input table is  $\emptyset$ , write  $\emptyset$  to the output tape;
    If row  $t$  column  $e$  of the input table is  $(s, f)$  with  $f \in \{(12), (23), (31)\}$ , write  $s' f$  to the output tape;
    If row  $t$  column  $e$  of the input table is  $(s, f)$  with  $f \in \{(21), (32), (13)\}$ , write  $s f$  to the output tape;
  end
end
for  $t \in [n]$  do
  Write # to the output tape;
  for  $e \in \{(12), (23), (31)\}$  do
    If row  $t$  column  $e$  of the input table is  $\emptyset$ , write  $\emptyset$  to the output tape;
    If row  $t$  column  $e$  of the input table is  $(s, f)$  with  $f \in \{(12), (23), (31)\}$ , write  $s f$  to the output tape;
    If row  $t$  column  $e$  of the input table is  $(s, f)$  with  $f \in \{(21), (32), (13)\}$ , write  $s' f$  to the output tape;
  end
end

```

that maps vertex  $i$  of triangle  $t$  to vertex  $j$  of triangle  $s$ . For the punctured Klein bottle example, this edge set is

$$\begin{aligned}
 F = \{ & \{w_{1,1}, w_{2,1}\}, \{w_{1,2}, w_{2,3}\}, \\
 & \{w_{2,1}, w_{3,1}\}, \{w_{2,2}, w_{3,3}\}, \\
 & \{w_{1,1}, w_{3,2}\}, \{w_{1,3}, w_{3,3}\}, \\
 & \{w_{1,2}, w_{3,1}\}, \{w_{1,3}, w_{3,2}\} \}.
 \end{aligned}$$

Algorithm 5 constructs this graph in logspace, simply by walking through the gluing table for the input triangulation. Note that, as it is presented here, Algorithm 5 writes each edge to the output tape twice; if desired this can easily be avoided using a lexicographical test.

**Algorithm 5:** Construct the vertex identification graph  $K = (W, F)$ .**Input:** Triangulation data on input tape.**Output:** The graph  $K = (W, F)$ .

```

for  $t \in [n]$  do
  for  $i \in \{1, 2, 3\}$  do
    Write  $w_{t,i}$  to the output tape;
  end
end
for  $t \in [n]$  do
  for  $e = (ij) \in \{(12), (23), (31)\}$  do
    Read the entry  $y = (s, (pq))$  in row  $t$  column  $e$ ;
    If  $y \neq \emptyset$ , write  $(w_{t,i}, w_{s,p})$  and  $(w_{t,j}, w_{s,q})$  to the output tape;
  end
end

```

Two vertices of  $K$  are in the same connected component of  $K$  if and only if the corresponding triangle vertices are identified in the input triangulation, and so  $|V|$  is the number of connected components of the graph  $K$ . Algorithm 1 with input  $K = (W, F)$  computes this number, and from this we can now compute the Euler characteristic  $\chi(S)$  of the input surface.

**4.3. Number of boundary components.** To count the number of boundary components in our surface, we build another auxiliary graph  $K'$ , which we call the *boundary identification graph*. This begins with the vertex identification graph  $K$ , and introduces additional edges that join together different paths in  $K$  that correspond to vertices on the same boundary component of the surface.

More precisely, this graph  $K'$  has vertex set  $W' = W$  as described above. The edge set of  $K'$  is

$$F' = F \cup \{(w_{t,i}, w_{t,j}) \mid \text{edge } (ij) \text{ of triangle } t \text{ is not glued to anything}\}.$$

For the punctured Klein bottle example, this edge set is

$$\begin{aligned} F' = \{ & \{w_{1,1}, w_{2,1}\}, \{w_{1,2}, w_{2,3}\}, \\ & \{w_{2,1}, w_{3,1}\}, \{w_{2,2}, w_{3,3}\}, \\ & \{w_{1,1}, w_{3,2}\}, \{w_{1,3}, w_{3,3}\}, \\ & \{w_{1,2}, w_{3,1}\}, \{w_{1,3}, w_{3,2}\}, \\ & \{w_{2,2}, w_{2,3}\} \}. \end{aligned}$$

Algorithm 6 shows how the graph  $K'$  is constructed.

<b>Algorithm 6:</b> Construct the boundary identification graph $K' = (W', F')$ .	
<b>Input:</b>	Triangulation data on input tape.
<b>Output:</b>	The graph $K' = (W', F')$ .
<b>for</b> $t \in [n]$ <b>do</b>	
<b>for</b> $i \in \{1, 2, 3\}$ <b>do</b>	
Write $w_{t,i}$ to the output tape;	
<b>end</b>	
<b>end</b>	
<b>for</b> $t \in [n]$ <b>do</b>	
<b>for</b> $e = (ij) \in \{(12), (23), (31)\}$ <b>do</b>	
Read the entry $y = (s, (pq))$ in row $t$ column $e$ ;	
If $y \neq \emptyset$ , write $(w_{t,i}, w_{s,p})$ and $(w_{t,j}, w_{s,q})$ to the output tape;	
If $y = \emptyset$ , write $(w_{t,i}, w_{t,j})$ to the output tape;	
<b>end</b>	
<b>end</b>	

We can analyse the structure of the vertex identification graph  $K$  and the boundary identification graph  $K'$ :

- $K$  is a disjoint union of cycles and paths, with one cycle for each internal vertex of the surface, and one path for each boundary vertex of the surface.
- $K'$  is a disjoint union of cycles, with one cycle for each internal vertex of the surface, and one cycle for each boundary component of the surface.

Moreover, the number of boundary vertices of the surface is equal to the number of boundary edges; that is, the number of  $\emptyset$  symbols in the gluing table for the input triangulation. Therefore counting boundary components becomes a simple matter of counting boundary edges and counting components of  $K$  and  $K'$ . Algorithm 7 gives the details.

<b>Algorithm 7:</b> Count the number of boundary components.	
<b>Input:</b>	Triangulation data on input tape.
<b>Output:</b>	Number of boundary components, $b$ .
Compose Algorithms 5 and 1 to find $k$ , the number of connected components of $K$ ;	
Compose Algorithms 6 and 1 to find $k'$ , the number of connected components of $K'$ ;	
Count the number of $\emptyset$ symbols on the input tape, and store this as the integer $x$ ;	
Write $b = k' - k + x$ to the output tape;	

We summarise this section in the following statement.

**Proposition 4.1.** *There is a logspace algorithm, Algorithm B, which given a triangulation of a connected surface  $S$  as input, computes  $(o, \chi, b)$  where  $o = 0$  if  $S$  is orientable and 1 if nonorientable,  $\chi = \chi(S)$  is the Euler characteristic of  $S$ , and  $b$  is the number of boundary components of  $S$ .*



## 5. MORE THAN ONE CONNECTED COMPONENT

We now assume the output to Algorithm 1 is  $c > 1$ . We will compute the following data:

$$(o_1, b_1, \chi_1), \dots, (o_c, b_c, \chi_c),$$

where  $o_i = 0$  if the  $i$ -th connected component is orientable and 1 otherwise,  $b_i$  is its number of boundary components, and  $\chi_i$  is its Euler characteristic. Moreover, we output this data in the following order:

- $o_i < o_{i+1}$ , or
- $o_i = o_{i+1}$  and  $\chi_i < \chi_{i+1}$ , or
- $o_i = o_{i+1}, \chi_i = \chi_{i+1}$ , and  $b_i \leq b_{i+1}$ .

The pseudocode is shown below, and followed by a discussion of the meta-algorithm.

**Algorithm 8:** Outputs the triangulation data for the  $i$ -th connected component only.

**Input:** Triangulation data on input tape with  $n$  triangles; integer  $i \leq n$   
**Output:** Triangulation data for the connected surface which is the  $i$ -th connected component of the input surface.  
 Initialise counters  $t = 1$  and  $c = 1$  (in binary) on the work tape;  
**while**  $c < i$  **do**  
   Increment  $t$  by 1;  
   Set  $b = \text{false}$ ;  
   **for**  $s = 1$  **to**  $t - 1$  **do**  
     If  $\text{REIN}([n], E, s, t)$  returns **true**, set  $b = \text{true}$ ;  
   **end**  
   If  $b = \text{false}$ , increment  $c$  by 1;  
**end**  
**for**  $s = t$  **to**  $n$  **do**  
   **if**  $\text{REIN}([n], E, s, t)$  returns **true** **then**  
     Write  $\#$  to the output tape;  
     **for**  $e \in \{(12), (23), (31)\}$  **do**  
       Read the entry  $y = (u, f)$  in row  $s$  column  $e$ ;  
       **if**  $y = \emptyset$  **then**  
         Write  $\emptyset$  to the output tape;  
       **end**  
       **else**  
         Initialise counter  $u' = 0$ ;  
         **for**  $x = t$  **to**  $s$  **do**  
           If  $\text{REIN}([n], E, x, t)$  returns **true** then increment  $u'$  by 1;  
         **end**  
         Write  $(u', f)$  to the output tape;  
       **end**  
     **end**  
   **end**  
**end**

Here is the meta-algorithm. Assume we have checked that the input is a surface, computed the number of triangles  $n$  and counted the number of connected components  $c > 1$ . Then using the algorithms described above, we compute the number of boundary components  $b$  and Euler characteristic  $\chi(S)$  for the entire (disconnected) surface  $S$ . Note that the number of boundary components (and connected components) is at most  $n$ .

The Euler characteristic for a connected surface is at most 2. We compute a lower bound on  $\chi$  for each connected component as follows. If  $S = \cup_{i=1}^c S_i$  are the connected components, we have  $\chi(S) = \sum_{i=1}^c \chi(S_i)$  so for one component we have  $\chi(S_{i_0}) = \chi(S) - \sum_{i \neq i_0} \chi(S_i)$ . This is minimised when the negative term on the right is maximised, and since the maximum Euler characteristic for any connected surface is 2, we have  $\chi(S_{i_0}) \geq \chi(S) - 2(c - 1)$ .

We then enumerate through all possible triples  $(o, x, \chi)$  for  $o \in \{0, 1\}$ ,  $0 \leq x \leq b$ ,  $\chi(S) - 2(c-1) \leq \chi \leq 2$  (which is a finite list), in the order given above. For each such triple  $(o, \chi, x)$ , we run through each connected component of  $S$  and compute  $(o_i, \chi_i, b_i)$ , and if  $(o_i, \chi_i, b_i) = (o, \chi, x)$  then we write this triple to the output.

This meta-algorithm repeatedly uses Algorithms A and B above. It also requires a logspace algorithm for extracting the  $i$ -th connected components of the input surface. We present such a procedure in Algorithm 8 which takes as input an integer  $i$  and triangulation data for a surface with possibly many connected components, and outputs the triangulation data for only the  $i$ -th connected component.

Algorithm 8 is a straightforward extension of Algorithm 1 (which just counts connected components). We draw attention to the final loop over the counter  $x$ , which is used to reindex the triangles on the output tape so that they are numbered consecutively as  $1, 2, \dots, k$ , where  $k$  is the number of triangles in the  $i$ -th component.

## 6. CONCLUDING REMARKS

The main result of this paper is in fact stronger than presented in the statement of Theorem 1.1 (that 2-manifold recognition is in L): the proof gives a logspace algorithm for the *function problem* to compute the homeomorphism type (essentially a “normal form”) of a given 2-manifold. This is in contrast to problems on some groups, such as braid groups with at least four strands, where there is a logspace solution to the word problem [4, 11, 12] but no known logspace algorithm for computing a normal form.

The only essential use of Reingold’s  $s, t$  connectivity algorithm in our work is in counting components of the surface, and testing orientability. In particular, the auxiliary graphs  $K$  and  $K'$ , which we use to compute Euler characteristic and count boundary components, have a very simple structure, and it is straightforward (but a little messier) to design custom logspace algorithms for counting their components that do not rely on REIN as an oracle. This raises the possibility that for connected and orientable surfaces, homeomorphism testing may be even simpler—for instance, an NC<sup>1</sup> solution might be possible.

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